Trigonometric Functions

Which Angle is Up?

Lesson 32-1 Placing the Unit Circle on the Coordinate Plane

ACTIVITY 32

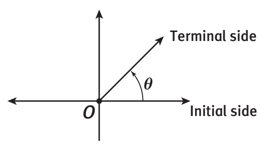
My Notes

Learning Targets:

- Explore angles drawn in standard position on the coordinate plane.
- Find the sine of θ and the cosine of θ.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Close Reading, Create Representations, Sharing and Responding, Look for a Pattern

In the last lesson you worked with angles formed by radii within a circle. In trigonometry, we work with angles on the coordinate plane. An angle is in ***standard position*** when the vertex is placed at the origin and the ***initial side*** is on the positive x-axis. The other ray that forms the angle is the ***terminal side***.

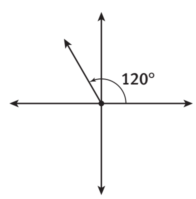


The terminal sides of angles with positive measures are formed by counterclockwise rotations. Angles with negative measures are formed by clockwise rotation of the terminal side.

Example A

Draw an angle in standard position with a measure of 120°.

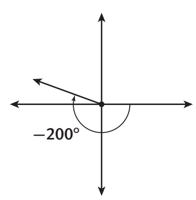
Since 120° is 30° more than 90°, the terminal side is 30° counterclockwise from the positive y-axis.



Example B

Draw an angle in standard position with a measure of −200°.

Since −200° is negative, the terminal side is 200° clockwise from the positive x-axis.



Common Core State Standards for Activity 32

HSF-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

ACTIVITY 32

Directed

Activity Standards Focus

Students have learned to calculate trigonometric ratios for acute angles using the ratios of the sides of a right triangle. In this activity, students will use reference angles and the unit circle to find trigonometric ratios of any angle. It is important that students understand angle measure expressed in both degrees and radians.

Lesson 32-1

PLAN

Pacing: 1 class period

Chunking the Lesson

Examples A–C #1
Example D
Check Your Understanding
Examples E–H
Examples I–J
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to convert degree measure to radian measure or radian measure to degree measure.

1. 135° $\left[\frac{3\pi}{4}\right]$
2. 480° $\left[\frac{8\pi}{3}\right]$
3. $\frac{7\pi}{6}$ [210°]
4. $\frac{11\pi}{3}$ [660°]

Developing Math Language

This lesson presents a number of new terms, as well as a review of vocabulary, including *radian*, *sine*, *cosine*, and *tangent*, from Activity 31 and from geometry. Have students sketch a picture in their journal to illustrate these terms. Remind students to add new terms and their definitions to their math journals. Then have students add these terms to the Interactive Word Wall.

Example A, Example B Create Representations Encourage students to sketch each angle as you discuss its measure. Model the process step by step. Sketch the initial side first. Discuss whether the angle is formed by clockwise or counterclockwise rotation and draw an arrow to indicate the appropriate direction. After approximating its location, sketch the terminal side of the angle. If possible, use different colors to emphasize the different components of each angle.

Example C Create Representations, Look for a Pattern, Debriefing Continue to model the process of sketching an angle in standard position. Invite students to discuss their strategies for understanding radian measure.

Universal Access

Throughout the lesson, encourage students to practice using radian measure. With each coordinate plane sketch, have students label the quadrantal angles $\left(\frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 0 \text{ or } 2\pi\right)$ for reference. Guide students to use what they know about fractions to help them label the terminal side of angles correctly.

CONNECT TO AP

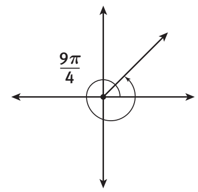
Functions in calculus use radian measure exclusively. It is imperative that students become comfortable with radian measure.

1 Sharing and Responding Invite students to share their personal strategies for understanding and identifying coterminal angles.

Example C

Draw an angle in standard position with a measure of $\frac{9\pi}{4}$ radians.

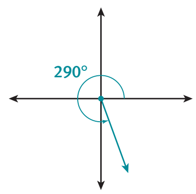
Since $\frac{9\pi}{4}$ is greater than 2π radians, the terminal side makes one full rotation, plus an additional $\frac{\pi}{4}$ radians.



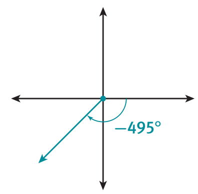
Try These A–C

Draw an angle in standard position with the given angle measure.

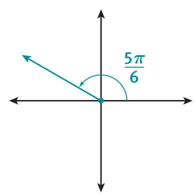
a. 290°



b. −495°



c. $\frac{5\pi}{6}$



Angles can have different rotations but have the same initial and terminal sides. Such angles are ***coterminal angles***. In Example C, you can see that an angle that is $\frac{9\pi}{4}$ radians is coterminal with an angle that is $\frac{\pi}{4}$ radians.

1. How can you find an angle that is coterminal with a given angle, whether given in degrees or in radians?
To find coterminal angles, add or subtract multiples of 360° or 2π radians.

Lesson 32-1
Placing the Unit Circle on the Coordinate Plane

My Notes

Example D
Find one positive and one negative angle that are coterminal with each given angle.

a. $225°$

$225° + 360° = 585°$

$225° - 360° = -135°$

b. $\frac{\pi}{3}$ radians

$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$

Try These D
Find one positive and one negative angle that are coterminal with each given angle. **Sample answers given.**

a. $150°$
$510°; -210°$

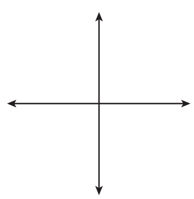
b. $320°$
$680°; -40°$

c. $-270°$
$90°; -630°$

d. $\frac{2\pi}{5}$ **$\frac{12\pi}{5}, -\frac{8\pi}{5}$**

Check Your Understanding

2. Draw an angle in standard position with a measure of $\frac{10\pi}{3}$ radians.



3. Find one positive and one negative angle that are coterminal with each of the given angles.
a. $-330°$ **b.** $480°$ **c.** $\frac{3\pi}{2}$

4. Are $520°$ and $-560°$ coterminal angles? Explain your answer.

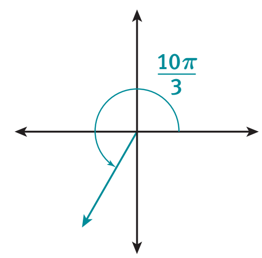
5. Are $\frac{10\pi}{6}$ and $-\frac{28\pi}{6}$ coterminal angles? Explain your answer.

6. Is there a limit to the number of coterminal angles an angle can have? Explain.

Example D Look for a Pattern, Debriefing Point out that there is more than one correct answer for each task. There are, in fact, an infinite number of angles that are coterminal with a given angle.

Check Your Understanding
Debrief students' answers to these items to ensure that they understand how to sketch an angle in standard position. Students should be comfortable with positive and negative angle measure in both radians and degrees. Monitor students' work to confirm that they can identify coterminal angles.

Answers
2.



3. Sample answers:
a. $30°; -690°$
b. $120°; -240°$
c. $\frac{7\pi}{2}, -\frac{\pi}{2}$

4. Yes; $520° - 360° - 360° - 360° = -560°$

5. No; when you subtract multiples of 2π from $\frac{10\pi}{6}$ you cannot obtain $-\frac{28\pi}{6}$ as an answer.

6. No; There are an infinite number of multiples of $360°$, or 2π radians, that you can add to or subtract from a given angle.

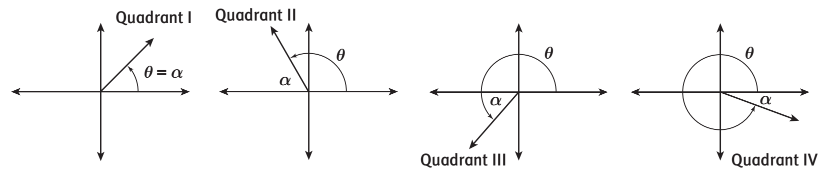
Paragraph Vocabulary Organizer, Close Reading, Note Taking When students add the definition of *reference angles* to their math journals, they should include one reference angle sketch for each quadrant.

Example E, Example F, Example G Close Reading, Create Representations, Debriefing Guide students to read each example carefully, paying attention to the purpose of each piece of information provided. Encourage students to sketch the angles as they work. Provide students with additional examples, as needed.

Lesson 32-1
Placing the Unit Circle on the Coordinate Plane

My Notes

If θ is an angle in standard position, its ***reference angle*** α is the acute angle formed by the terminal side of θ and the x-axis. The graphs show the reference angle α for four different angles that have their terminal sides in different quadrants.



The relationship between θ and α is shown for each quadrant when $0° < \theta < 360°$ or $0 < \theta < 2\pi$.

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Degrees:	$\alpha = \theta$	$\alpha = 180° - \theta$	$\alpha = \theta - 180°$	$\alpha = 360° - \theta$
Radians:	$\alpha = \theta$	$\alpha = \pi - \theta$	$\alpha = \theta - \pi$	$\alpha = 2\pi - \theta$

Example E

Find the reference angle for $\theta = 245°$.

The terminal side of θ lies in Quadrant III.

$\alpha = 245° - 180°$, so $\alpha = 65°$.

Example F

Find the reference angle for $\theta = \frac{3\pi}{4}$.

The terminal side of θ lies in Quadrant II.

$\alpha = \pi - \frac{3\pi}{4}$, so $\alpha = \frac{\pi}{4}$.

When an angle is not between 0 and $360°(2\pi)$, find a coterminal angle that is within that range. Then use the coterminal angle to find the reference angle.

Example G

Find the reference angle for $\theta = 435°$.

Since $435°$ is greater than $360°$, subtract.

$435 - 360 = 75°$

Now determine the reference angle for $75°$.

Since $75°$ is in Quadrant I, the reference angle is $75°$.

Lesson 32-1
Placing the Unit Circle on the Coordinate Plane

Example H
Find the reference angle for $\theta = \frac{13\pi}{4}$ radians.

Since $\frac{13\pi}{4}$ is greater than 2π, subtract.

$\frac{13\pi}{4} - 2\pi = \frac{13\pi}{4} - \frac{8\pi}{4} = \frac{5\pi}{4}$

The terminal side of this angle is in Quadrant III.

$\alpha = \frac{5\pi}{4} - \pi = \frac{5\pi}{4} - \frac{4\pi}{4}$, so $\alpha = \frac{\pi}{4}$.

Try These E–H
Find the reference angle for each value of θ.

a. $\theta = 325°$

$\alpha = 35°$

b. $\theta = \frac{4\pi}{9}$

$\alpha = \frac{4\pi}{9}$

c. $\theta = 515°$

$\alpha = 25°$

d. $\theta = \frac{7\pi}{6}$

$\alpha = \frac{\pi}{6}$

e. $\theta = \frac{10\pi}{3}$

$\alpha = \frac{\pi}{3}$

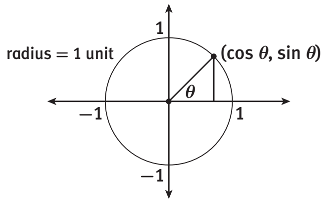
f. $\theta = 820°$

$\alpha = 80°$

Coterminal and reference angles can be used to find trigonometric functions for angles in standard form. ***Trigonometric functions*** describe the relationships between sides and angles in a triangle.

We can look at these relationships on the unit circle. Since the radius of the unit circle is 1 unit, any right triangle with a hypotenuse formed by this radius has a hypotenuse length of 1 unit.

Unit Circle



$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$

My Notes

MATH TIP
You learned the trigonometric functions for sine and cosine in geometry.

$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$

$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$

ACTIVITY 32 Continued

Example H Create Representations, Think-Pair-Share, Debriefing

Encourage students to sketch each angle as they work. Have students present examples of other angles that have $\frac{\pi}{4}$ as a reference angle. Challenge students to identify at least one (positive) angle and one negative angle from each quadrant.

ELL Support
If students still find radian measure challenging, allow them to convert from radian measure to degrees in order to solve the problem. Point out, however, that working with radians is like learning a new language. In the beginning, you may need to translate from radian measure to degree measure. Once fluent, though, a student should be able to solve problems easily using radians without the need to convert to the original measure.

Differentiating Instruction
Call on students to review their prior knowledge of trigonometric functions, perhaps with help from doing the mini-lesson. Use their current understanding to provide background information to help them comprehend the relationship between the unit circle and trigonometric functions.

MINI-LESSON: Trigonometry Review

If students need additional help with trigonometry, a mini-lesson is available to review basic definitions and concepts.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

Paragraph Close Reading, Create Representations, Debriefing The definition of cosine of θ and sine of θ as the x and y coordinates of a point on the unit circle is key to understanding how to determine the trigonometric functions of all angle measures. Draw students' attention to this representation of trigonometric values. Ask questions to ensure that students understand the extension of the trigonometric functions summarized in these statements.

Example I, Example J Create Representations, Think-Pair-Share, Debriefing These are the first examples showing how to find trigonometric functions of nonacute angles. While the trigonometric functions can be calculated for acute angles using right triangle ratios, the functions cannot be calculated for the quadrantal angles in the same manner.

Have students work together to solve the Try These items. Encourage students to share their strategies and solutions with the class.

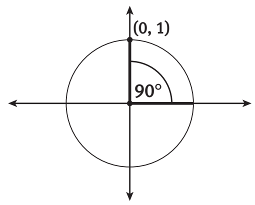
My Notes

Lesson 32-1
Placing the Unit Circle on the Coordinate Plane

The cosine of θ ($\cos\theta$) is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle. The sine of θ ($\sin\theta$) is the y-coordinate.

Example I

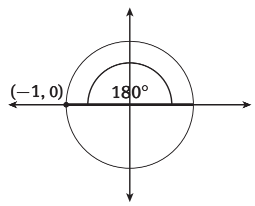
Find the sine and cosine of 90°.



$\sin 90° = 1$
$\cos 90° = 0$

Example J

Find the sine and cosine of 180°.



$\sin 180° = 0$
$\cos 180° = -1$

Try These I–J

a. What are the sin θ and cos θ for $\theta = 270°$, $\theta = -270°$, and $\theta = 720°$?

$\sin 270° = -1$ $\quad$ $\sin(-270°) = 1$ $\quad$ $\sin 720° = 0$

$\cos 270° = 0$ $\quad$ $\cos(-270°) = 0$ $\quad$ $\cos 720° = 1$

b. What are the sin θ and cos θ for $\theta = \pi$, $\theta = 2\pi$, and $\theta = -\frac{\pi}{2}$?

$\sin \pi = 0$ $\quad$ $\sin 2\pi = 0$ $\quad$ $\sin\left(-\frac{\pi}{2}\right) = -1$

$\cos \pi = -1$ $\quad$ $\cos 2\pi = 1$ $\quad$ $\cos\left(-\frac{\pi}{2}\right) = 0$

Lesson 32-1
Placing the Unit Circle on the Coordinate Plane

My Notes

Check Your Understanding

7. Find the reference angle for each value of θ.
a. $\theta = 135$ **b.** $\theta = 240°$
c. $\theta = \frac{7\pi}{6}$ **d.** $\theta = \frac{5\pi}{3}$

8. Find the value of $\sin\theta$ and $\cos\theta$ for each angle.
a. $\theta = 360°$ **b.** $\theta = -90°$ **c.** $\theta = -\frac{7\pi}{2}$

LESSON 32-1 PRACTICE

9. Draw an angle in standard position with a measure of $-\frac{7\pi}{3}$ radians.

10. Give one positive and one negative angle that are coterminal with $-390°$.

11. What is the reference angle for each value of θ?
a. $\theta = \frac{17\pi}{6}$ **b.** $\theta = -250°$?

12. What are the sine and cosine for each value of θ?
a. $\theta = 270°$ **b.** $\theta = -5\pi$

13. Attend to precision. Refer to Examples I and J and Try These I–J. Do you notice anything about the sine and cosine of angles that are multiples of 90°?

Check Your Understanding
Debrief students' answers to these items to ensure that they understand how to identify reference angles. Students should be able to use reference angles to calculate the sine and cosine of any angle.

Answers

7. a. 45°
b. 60°
c. $\frac{\pi}{6}$
d. $\frac{\pi}{3}$

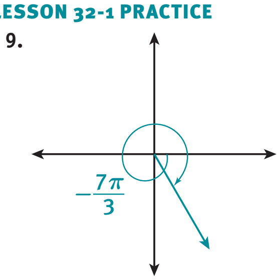
8. a. 0, 1
b. -1, 0
c. 1, 0

ASSESS
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 32-1 PRACTICE

9.



10. Sample answers: $-30°$, 330°
11. a. $\frac{\pi}{6}$
b. 70°
12. a. -1, 0
b. 0, -1
13. The sine and cosine of each of these angles is either 0, 1, or -1.

ADAPT
Check students' answers to the Lesson Practice to ensure that they understand how to sketch an angle in standard position and identify the reference angle for this angle. Monitor students' progress to ensure that students understand how to use the unit circle to find the sine and cosine of any angle.

Lesson 32-2

PLAN

Pacing: 1–2 class periods

Chunking the Lesson

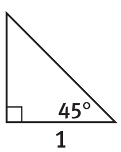
Example A Example B
Examples C–D
Check Your Understanding
#4 #5–6
Check Your Understanding
Lesson Practice

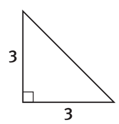
TEACH

Bell-Ringer Activity

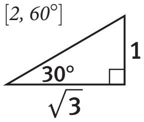
Ask students to find the length of each hypotenuse and the measure of each angle.

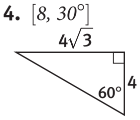
1. [$\sqrt{2}$, 45°] **2.** [$3\sqrt{2}$, 45°]





3. [2, 60°] **4.** [8, 30°]





Teacher to Teacher

Monitor students' work on the Bell-Ringer Activity to ensure that they understand how the side lengths of the two standard triangles are calculated. Review the relationships among sides and angles of special right triangles. Lead students in a discussion of how they can use their knowledge of special right triangles to identify trigonometric values of the special angles. Remind students of this discussion when you discuss the unit circle.

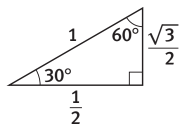
Example A Create Representations Encourage students to draw a picture as they evaluate sine and cosine. Ask students which is greater: $\sqrt{3}$ or 1. Invite students to share how they can use this information to help them find the sine and cosine of 30° using a 30°-60°-90° triangle.

Example B Create Representations Encourage students to draw the triangle formed by the unit circle showing the reference angle for $\frac{7\pi}{4}$ radians. Use the orientation of the triangle to help students determine whether the values of sine and cosine are positive or negative.

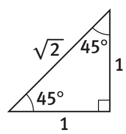
My Notes

MATH TIP

The ratio of the side lengths of a 30°-60°-90° triangle is $1 : \sqrt{3} : 2$, and of a 45°-45°-90° triangle is $1 : 1 : \sqrt{2}$. If the length of the hypotenuse of a 30°-60°-90° triangle is equal to 1, then the ratio must be divided by 2 to find the lengths of the legs, $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$.



If the length of the hypotenuse of a 45°-45°-90° triangle is 1, then the ratio must be divided by $\sqrt{2}$ to find the length of both legs, $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.



Learning Targets:

- Find the sine of θ and the cosine of θ using special right triangles.
- Find the tan of θ.

SUGGESTED LEARNING STRATEGIES: Create Representations, Graphic Organizer, Look for a Pattern

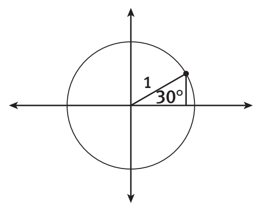
You can use what you know about the ratios of side lengths of special right triangles to determine the sine and cosine of their angles. As shown in the last lesson, a right triangle whose hypotenuse is a radius of the unit circle has a hypotenuse length of 1 unit. The hypotenuse is the terminal side of an angle, θ, and the sine and cosine of θ are the lengths of the legs of the right triangle.

Example A

What are the sine and cosine of θ?

$\theta = 30°$

The sine and cosine are the lengths of the legs of a 30°-60°-90° triangle.



$\sin 30° = y = \text{length of shorter leg} = \frac{1}{2}$

$\cos 30° = x = \text{length of longer leg} = \frac{\sqrt{3}}{2}$

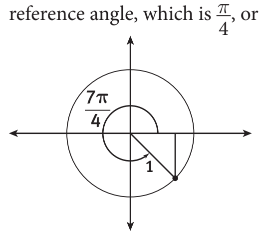
If θ is not in the first quadrant, use a reference angle.

Example B

What are $\sin\theta$ and $\cos\theta$?

$\theta = \frac{7\pi}{4}$ radians

To find $\sin\theta$ and $\cos\theta$, draw the terminal side of the angle on the unit circle. Make a right triangle with one leg on the x-axis. Determine the reference angle, which is $\frac{\pi}{4}$, or 45°. The triangle is a 45°—45°—90° triangle.



$\sin\frac{7\pi}{4} = y = -\text{length of opposite leg} = -\frac{\sqrt{2}}{2}$

$\cos\frac{7\pi}{4} = x = \text{length of adjacent leg} = \frac{\sqrt{2}}{2}$

Lesson 32-2
Special Right Triangles and the Unit Circle

My Notes

Try These A–B
What are $\sin\theta$ and $\cos\theta$ for each value of θ?

a. $\theta = 300°$ $\quad -\frac{\sqrt{3}}{2}, \frac{1}{2}$

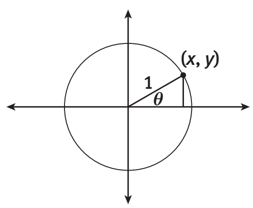
b. $\theta = -225°$ $\quad \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

c. $\theta = \frac{5\pi}{6}$ $\quad \frac{1}{2}, -\frac{\sqrt{3}}{2}$

d. $\theta = -\frac{4\pi}{3}$ $\quad \frac{\sqrt{3}}{2}, -\frac{1}{2}$

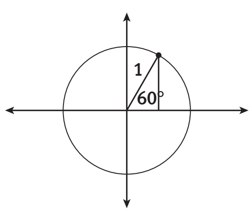
Sine and cosine are just two of the trigonometric functions. Next we will look at a third function, the tangent function.

Recall that the tangent function for a right triangle is $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$. Looking at the unit circle on the coordinate plane, you can see that this can also be expressed as $\tan\theta = \frac{y}{x}$, where y and x are the coordinates at the point of intersection of the terminal side of θ and the unit circle.



As with the relationships we saw with sine and cosine, this relationship is also true for all angles on the unit circle.

Example C
What is $\tan\theta$ for $\theta = 60°$?



$$\tan 60° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Technology Tip
Students may want to use calculators to check their work. Where possible, set the calculator mode to "Exact Value." This allows students to see the exact value of the answer in fraction form rather than as a decimal approximation.

If students are struggling to verify an answer, check to ensure that their calculator is using the correct angle measure, radians or degrees.

For additional technology resources, visit SpringBoard Digital.

ELL Support
Some students may confuse *sine* and *sign* as they work with trigonometric functions. Remind students that *sine* is the name of a trigonometric function defined by a ratio of sides of a right triangle. The *sine* of a given angle may have a positive or negative *sign*.

Try These A–B Look for a Pattern Ask students to compare their answers for Items a and d. How are the two angles, 300° and $-\frac{4\pi}{3}$, alike? [*They have the same reference angles, so their sines and their cosines have equal absolute values.*] How are they different? [*Their terminal sides lie in different quadrants.*] How can you use these observations to help you solve problems in the future?

Example C Close Reading, Look for a Pattern Students review a third trigonometric function—tangent—over the set of all real numbers. Draw students' attention to the importance of the last sentence before this example. Use the unit circle to point out the following relationships:

$$\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\sin\theta}{\cos\theta}$$

Encourage students to use this relationship to minimize the amount of memorization needed to evaluate the trigonometric functions at key values.

Example D Look for a Pattern, Debriefing Ask students questions to help them understand the sign of trigonometric functions. How do you determine the sign of the trigonometric value? What patterns did you use to answer this question?

Point out the Math Tip. Ask students to identify several values of θ for which $\tan\theta$ is undefined.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to use the unit circle to evaluate sine, cosine, and tangent for a standard angle. Students should be equally comfortable working with degrees or radians. Encourage students to draw a sketch of each angle to help them identify its trigonometric values.

Answers

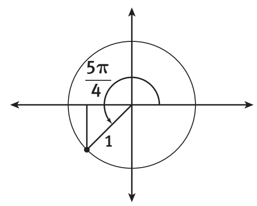
1. a. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$
   b. $\frac{\sqrt{3}}{2}, -\frac{1}{2}$
   c. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$
2. a. $\sqrt{3}$
   b. $-\frac{\sqrt{3}}{3}$
   c. -1
3. a. $\frac{\sqrt{3}}{3}$
   b. $\sqrt{3}$
   c. -1

My Notes

Example D

What is $\tan\theta$ for $\theta = \frac{5\pi}{4}$?

Use the reference angle $\frac{\pi}{4}$.



$$\tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Try These C–D

Find $\tan\theta$ for each value of θ.

a. $\theta = 300°$ $-\sqrt{3}$

b. $\theta = 450°$ undefined

c. $\theta = \frac{2\pi}{3}$ $-\sqrt{3}$

d. $\theta = \frac{11\pi}{4}$ -1

MATH TIP

When a ratio has a denominator of 0, the ratio is *undefined*.

MATH TIP

When a ratio has an irrational number in the denominator, the denominator needs to be rationalized.

Multiply the numerator and denominator by the irrational number.

For example, $\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check Your Understanding

1. Find $\sin\theta$ and $\cos\theta$.
   a. $\theta = 210°$ b. $\theta = \frac{2\pi}{3}$ c. $\theta = -\frac{\pi}{4}$
2. Find $\tan\theta$ for each value of θ.
   a. $\theta = 240°$ b. $\theta = 690°$ c. $\theta = -585°$
3. What is $\tan\theta$ for these values of θ?
   a. $\theta = \frac{7\pi}{6}$ b. $\theta = \frac{7\pi}{3}$ c. $\theta = -\frac{9\pi}{4}$

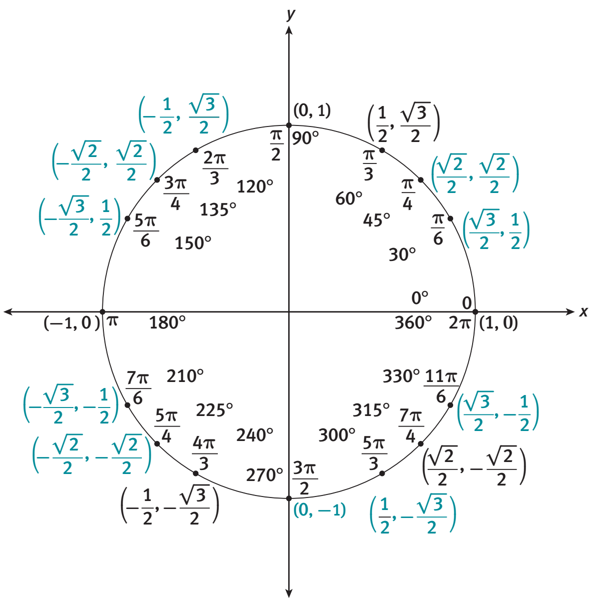
Lesson 32-2
Special Right Triangles and the Unit Circle

My Notes

The terminal side of every angle in standard position has a point that intersects the unit circle. You have seen that a right triangle can be drawn with the terminal side of each angle as the hypotenuse. One leg of the triangle is the segment drawn from the point of intersection to the *x*-axis, and the other leg is the segment of the *x*-axis from the origin to the point of intersection with the vertical segment.

You have been looking at 30°-60°-90° triangles and 45°-45°-90° triangles. All of the angles that can form these two triangles are given on the unit circle below in degrees and radians.

4. Use the reference angle that can be formed to find the *x*- and *y*-coordinates for each point of intersection on the unit circle.



As you have seen in Lesson 32-1 and in the first part of this lesson, you can find the values of the trigonometric functions sine, cosine, and tangent using the coordinates of the point of intersection of the terminal side of each angle with the unit circle.

5. Use the coordinates you found in Item 4. What are the sine, cosine, and tangent of 210°?

$\sin 210° = -\frac{1}{2}$, $\cos 210° = -\frac{\sqrt{3}}{2}$, $\tan 210° = \frac{\sqrt{3}}{3}$

6. What are the sine, cosine, and tangent of $\frac{5\pi}{4}$ radians?

$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $\tan\frac{5\pi}{4} = 1$

MATH TIP

The coordinates of the intersection of the terminal side of an angle θ with the unit circle are $(\cos\theta, \sin\theta)$.

Paragraph Close Reading, Note Taking, Marking the Text Draw a unit circle as you discuss the concepts in this paragraph. Draw a 30° angle in standard position so that the terminal side of the angle intersects the unit circle. Plot the point of intersection. Invite a student to sketch the right triangle described in the paragraph and to identify its legs and hypotenuse. Ask students to find the sine and cosine of the 30° angle and use these values to label the plotted point. Next, follow these steps to plot and label the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

4 Look for a Pattern, Think-Pair-Share, Debriefing This diagram of the unit circle will be very important as students move forward through trigonometry, precalculus, and calculus. Encourage students to use patterns rather than rote memorization to understand and remember the values included in this sketch of the unit circle.

5–6 Look for a Pattern, Debriefing Provide additional practice for students, as needed. Encourage students to use reference angles to understand each solution.

TEACHER to TEACHER

Many students may not understand sin and cos as the names of functions. Point out that $\sin\theta$ represents the sine of the angle θ, or the value of the trigonometric function, sine, evaluated at θ. Using $\sin\theta$ is analogous to labeling a value $f(x)$.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to evaluate the sine, cosine, and tangent of any angle. Invite students to share their strategies.

Answers

7. $\sin 495° = \frac{\sqrt{2}}{2}$, $\cos 495° = -\frac{\sqrt{2}}{2}$, $\tan 495° = -1$
8. $\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos \frac{5\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan \frac{5\pi}{4} = -1$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 32-2 PRACTICE

9. a. $\frac{\sqrt{3}}{2}, \frac{1}{2}$
   b. $\frac{\sqrt{3}}{2}, -\frac{1}{2}$
10. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
11. (0, 1)
12. a. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$
    b. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$
    c. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1$
13. a. $-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1$
    b. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}$
    c. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}$
14. Yes; if sine and cosine are both negative, then tangent will be positive. This happens in Quadrant III.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand basic concepts related to angles in standard position and in the unit circle, as well as for finding the sine, cosine, and tangent of any angle.

My Notes

Check Your Understanding

7. What are the sine, cosine, and tangent of 495°?
8. What are the sine, cosine, and tangent of $\frac{7\pi}{4}$ radians?

LESSON 32-2 PRACTICE

9. Find the sine and cosine for each value of θ.
   a. -300
   b. $\frac{8\pi}{3}$
10. What is the point of intersection of the terminal side of $-120°$ with the unit circle?
11. What is the point of intersection of the terminal side of $\frac{5\pi}{2}$ with the unit circle?
12. Find the sine, cosine, and tangent of each angle.
    a. 780°
    b. $-150°$
    c. $-405°$
13. Find the sine, cosine, and tangent of each angle.
    a. $-\frac{11\pi}{4}$
    b. $-\frac{7\pi}{6}$
    c. $\frac{10\pi}{3}$
14. **Make sense of problems.** Look at the unit circle in Item 4. Is it possible to have a negative value for sine but a positive value for tangent? Explain.

ACTIVITY 32 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 32-1

1. Draw an angle in standard position for each of the following measures.

a. $200°$ **b.** $575°$

c. $-225°$ **d.** $-660°$

e. $\frac{2\pi}{5}$ **f.** $-\frac{3\pi}{2}$

g. $-\frac{9\pi}{4}$ **h.** $\frac{11\pi}{3}$

2. Which angle is a coterminal angle with $140°$?
A. $-140°$ **B.** $40°$
C. $400°$ **D.** $500°$

3. Which angle is a coterminal angle with $-75°$?
A. $435°$ **B.** $-285°$
C. $285°$ **D.** -645

4. Which angle is *not* a coterminal angle with $\frac{5\pi}{4}$ radians?
A. $-\frac{3\pi}{4}$ **B.** $-\frac{7\pi}{4}$
C. $-\frac{11\pi}{4}$ **D.** $\frac{13\pi}{4}$

5. Give one positive and one negative angle that are coterminal with each of the following angles.
a. $-65°$ **b.** $500°$
c. $-\frac{6\pi}{5}$ **d.** $\frac{8\pi}{3}$

6. What is the reference angle for $\theta = 75°$?
A. $15°$ **B.** $75°$
C. $105°$ **D.** $255°$

7. What is the reference angle for $\theta = \frac{8\pi}{5}$?
A. $\frac{\pi}{5}$ **B.** $\frac{2\pi}{5}$
C. $\frac{3\pi}{5}$ **D.** $\frac{8\pi}{5}$

8. What is the reference angle for each value of θ?
a. $\theta = -325°$ **b.** $\theta = 530°$
c. $\theta = -\frac{12\pi}{5}$ **d.** $\theta = \frac{7\pi}{4}$

9. In which quadrant is the reference angle α equal to θ?

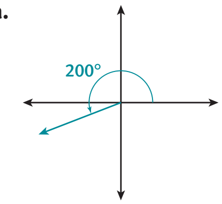
10. Find $\sin\theta$ and $\cos\theta$.
a. $\theta = -180°$ **b.** $\theta = 450°$?

11. Find $\sin\theta$ and $\cos\theta$.
a. $\theta = 6\pi$ **b.** $\theta = -\frac{7\pi}{2}$

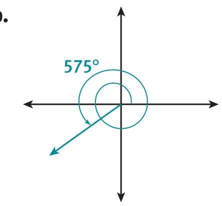
12. What are the sine and cosine for each value of θ?
a. $\theta = 315°$ **b.** $\theta = -510°$
c. $\theta = -\frac{11\pi}{6}$ **d.** $\theta = \frac{10\pi}{3}$

ACTIVITY PRACTICE

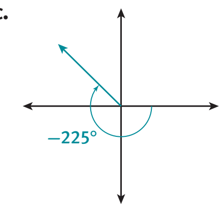
1. a.



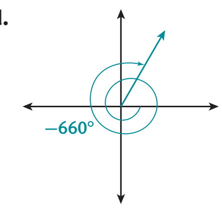
b.



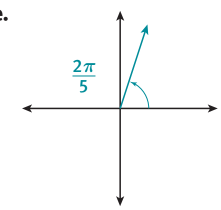
c.



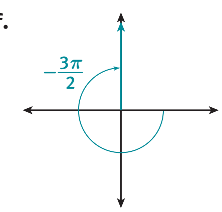
d.



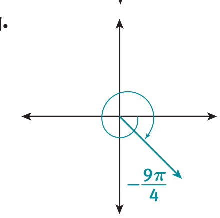
e.



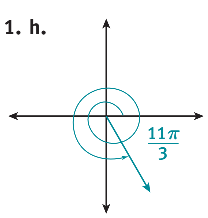
f.



g.



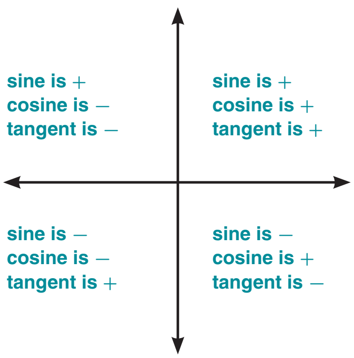
1. h.



2. D
3. C
4. B
5. Sample answers:
a. $-425°, 295°$
b. $140°, -220°$
c. $\frac{4\pi}{5}, -\frac{16\pi}{5}$
d. $\frac{2\pi}{3}, -\frac{4\pi}{3}$
6. B
7. B
8. a. $35°$
b. $10°$
c. $\frac{2\pi}{5}$
d. $\frac{\pi}{4}$
9. Quadrant I
10. a. $0, -1$
b. $1, 0$
11. a. $0, 1$
b. $1, 0$
12. a. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$
b. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$
c. $\frac{1}{2}, \frac{\sqrt{3}}{2}$
d. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$

ACTIVITY 32 Continued

13. D
14. B
15. D
16. **a.** 1
b. $-\frac{\sqrt{3}}{3}$
c. $-\sqrt{3}$
d. undefined
17. $225°$
18. $\frac{5\pi}{3}$
19. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$
20. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$
21. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$
22. $-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1$
23. Explanations will vary.



ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Lesson 32-2

13. What is $\tan \theta$ for $\theta = -300°$?
A. $-\frac{\sqrt{3}}{3}$ **B.** $\frac{\sqrt{3}}{2}$
C. $\frac{1}{2}$ **D.** $\sqrt{3}$

14. What is $\tan \theta$ for $\theta = \frac{19\pi}{6}$?
A. $-\sqrt{3}$ **B.** $\frac{\sqrt{3}}{3}$
C. $\frac{\sqrt{3}}{2}$ **D.** $-\frac{1}{2}$

15. What is $\tan \theta$ for $\theta = 765°$?
A. $\sqrt{2}$ **B.** $\frac{\sqrt{2}}{2}$
C. -1 **D.** 1

16. What is $\tan \theta$ for each value of θ?
a. $\theta = -495°$ **b.** $\theta = 690°$
c. $\theta = \frac{14\pi}{3}$ **d.** $\theta = -\frac{7\pi}{2}$

17. Give an angle measure in degrees, between $0°$ and $360°$, whose terminal side has a point of intersection with the unit circle at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

18. Give an angle measure in radians, between π and 2π, whose terminal side has a point of intersection with the unit circle at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

19. What are the sine, cosine, and tangent of $390°$?

20. What are the sine, cosine, and tangent of $-510°$?

21. What are the sine, cosine, and tangent of $\frac{13\pi}{3}$?

22. What are the sine, cosine, and tangent of $-\frac{11\pi}{4}$?

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

23. Use the unit circle in Item 4 of Lesson 32-2. Determine which trigonometric functions are positive and which are negative in each quadrant. Explain how you determined the signs for each quadrant. Summarize your findings on a coordinate plane like the one below.